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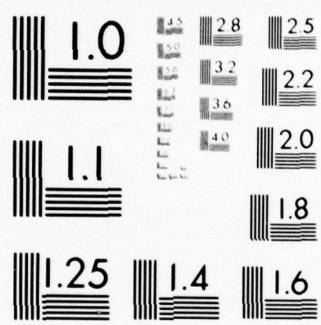

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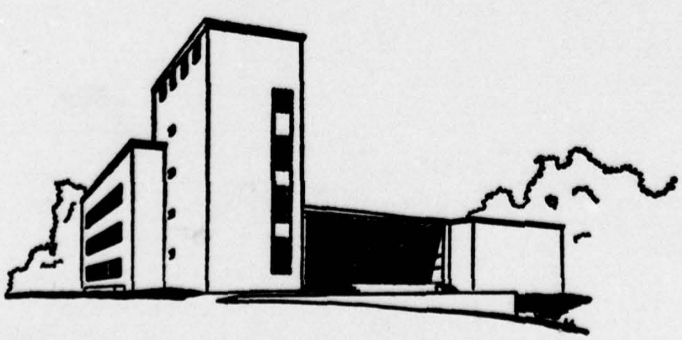

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Abstract

The paper presents the aggregation of heterogenous production into a homogenous one in an economy described by a General Equilibrium model of the von Neumann type (a KMT model with final demand). The resulting aggregated model will be of the von Neumann-Leontief type. We will consider two cases, one in which the final demand is given by a vector of parameters, and another case in which we will have a final demand function.

Key Words:

aggregation

expending economy model

von Neumann Model

input-output data

generalized inverses

1. Introduction

Firms that produce an homogenous output are a useful device for both empirical and theoretical research. It is obvious though, that we live in a world where the product of a firm in heterogenous, and the problem is how to reconcile the theory and the "reality." In this paper we do not attempt to give a general answer to this question, and we will restrict our analysis to the case in which the economy is linear.

Similar questions were raised in the economic literature especially connected with the input-output analysis. Since one of the main assumptions of the input-output analysis is that each firm produces a distinct homogenous output, it is clear that some aggregation of the heterogenous goods has to be performed in order to obtain the idealized image of the economy assumed by the Leontief model of general equilibrium. In general it has been admitted that the conditions for "good" aggregation are given only by very unrealistic conditions of complementarity, substitutability and similarities between firms.[1/] It should be mentioned that these studies implicitly dealt with a somewhat simplier problem. Namely, it was assumed that the actual economy presents the property of homogeneous production but its size is too big and the problem was how to reduce it to more manageable proportions. Now, the dimensionality problem in itself might not be any longer a problem (faster computers, etc.), but the fact that the firms have a heterogenous product it is still a problem.

In the first part of our paper we will present the results obtained through a standard approach to the aggregation problem. As it will be seen, these results do not differ very much from the conditions of substitutability and so on mentioned before. In the second part of the paper, by introducing more structure in our framework, we will approach the same problem in a different way. We will show that under this alternative approach the conditions for consistent aggregation are less restrictive.

In the remainder of this section we will outline the main assumptions of our approach. In order to simplify our presentation we will use terms like micromodel, microvariables whenever we refer to the model and variables that describe the economy in which the firms are producing a heterogenous output. Following the same convention, we refer to the economy in which each firm produces a homogenous output as the macroeconomy and respectively macromodel, macrovariables and so on.

For the microeconomy, we assume that the technical production possibilities of each firm are described by an activity analysis production function.[2/] In this way we allow joint production for a firm but we make our analysis simplier due to the constant returns to scale and the additivity properties possessed by this type of production function. Given this particular form of the production function, we further assume that by aggregating all the goods a given firm produces into one good (which we will call "product"), we obtain a new firm (which we will call "sector"), whose production possibilities are described by a new linear production function which differs from the firms production function in only one respect: the sector produces only one product which is unique, i.e. it is not produced by any other sector. It is clear then, that the production possibilities of each sector are described by a Leontief production function. It can be noted that up to here our approach follows the approach outlined by Klein.[3/] Namely, he required that the production function which contains aggregated data should be of the same form with the production functions from the micromodel. In the same paper, Klein required also that the maximizing behavior of the microfirms should be "transferred," through the aggregation function, to the macrofirms.[4] It is here that we add more structure to the approach to the aggregation problem; we assume that the firms are not only maximizing their profits for given prices, but more than that, we assume that the microeconomy is in equilibrium and so the prices are also determined. Given the form of the production function of the firms, the equilibrium in the microeconomy is then described by a general equilibrium model of the von Neumann variety, namely a KMT model.[5] It should be mentioned that the aggregation in KMT models was never studied before from this point of view.[6]

In order to follow in the spirit of Klein's approach, we require that the equilibrium in the macromodel should have the same characteristics as the equilibrium relations from the micromodel--the quantities produced by the sectors and the prices of the products should be determined such that the demand for every product is satisfied and the economy is growing uniformly and efficiently (using Koopman's term).[7] As said before, the same characteristics determine the equilibrium prices and quantities in the micromodel, and so they have to appear at the macro level too. Under these circumstances, we assume that the equilibrium in the macroeconomy is described by a von Neumann-Leontief model.[8] It should be mentioned that though the connections between various Leontief type models has been studied quite intensively, the connection between the KMT model and a von Neumann-Leontief (vNL) model has not been studied before.[9] For future reference, it should be mentioned that since each sector produces a unique product, the equilibrium conditions in the von Neumann-Leontief model describe the exchange that takes place between sectors under the assumption that the exchange between any two sectors is proportional to some fixed non-negative real number.

Because of our general equilibrium approach, we have to complete our model by specifying the consumption side of the microeconomy. Introducing consumption in a von Neumann type of model is not a trivial matter and it seems that the better insight this type of models give on the production side of the economy is paid with a rather formalistic approach to the consumption side of the economy.[10/] In our paper we consider two ways of introducing the public's demand for goods in our micromodel.

In the first case we simply assume that the demand for each good is some unknown non-negative real number, and so the demand of the whole economy is a vector of non-negative real parameters. Consequently, the demand for products will be also represented by a vector of parameters of a smaller dimension. Not unexpectedly, the conditions for consistent aggregation require the exchange that takes place among the firms of the KMT economy to have a very restrictive linear form. We argue that the strong conditions for consistent aggregation we obtained in this case are due to two factors: on one hand we require that the microvariables (the prices of the goods) and the microparameters (the demands for goods) to be aggregated in exactly the same way, and on the other hand we require the aggregation procedure to realize a consistent aggregation for a very large range of values of the demand.

In order to show that, in the second case, we will assume that the consumption side of the economy is described by a vector of functions and we will redefine accordingly the equilibrium conditions in the microeconomy. By taking advantage of this more structured framework, we will show that there are KMT models whose form of trading is other than linear but which can be aggregated to vNL model because their equilibrium values are <u>as if</u> the trading was linear.

The paper proceeds as follows. In section II we introduce our models and we study the first case in which the demand is given in parametric form. In section III we study the second case and in section IV we will present a comparison of these two cases. Section V presents our concluding remarks.

II. The Parametric Demand Case

In the first part of this section we introduce our models and some definitions, while in the second part we give the necessary and sufficient conditions for consistent aggregation for the case in which the final demand is a vector of parameters.

II.1 As mentioned before we assume that the firms of our microeconomy are in long run equilibrium and the economy is on a uniform rate of growth path--the prices and output are growing at a unique constant rate.

We assume that our microeconomy consists of m firms which are producing n goods, and joint production is possible in the economy $(m \leq n)$. The production takes place during an infinite number of time intervals of equal size and the economy is allowed to grow from one period to another at a constant rate. That means that the state of the economy in any future period is proportional to its state in the first period. Each good is characterized by a relative price, where the numeraire is such that the sum of all the relative prices is equal to one. Each firm is characterized by an intensity, where the intensity for a firm expresses the relative weight that particular firm has as compared to the other firms from the output point of view. The sum of all the intensities is equal to one.

These three variables has to be determined such that: the supply from every good exceeds the demand from that good (2), each firm has to make a negative or zero "profit" (3), in equilibrium the free goods (goods are which the supply exceeds the demand) have zero prices and the inefficient firms (firms that have strictly negative profits) are not run, i.e. their intensities are equal to zero (4). In order to obtain meaningful economic solutions, we will assume that over all the economy, something of value is produced (5).[11/]

As said before, for this case we will consider that the demand for goods is represented by a n-dimensional row vector of non-negative parameters. This amounts to assuming the demand for consumption goods is equal to the demand for saving goods.[12/] The equilibrium in the microeconomy is then summarized by the following relations (a KMT with final demand model):

(2)	$q(B-\alpha A^k) \geq \alpha d^k$	(supply exceeds demand)
(3)	$(B-\alpha A^k)p \leq \alpha f d^k p$	(profitless condition)
(4)	$q(B-\alpha A^k)p = \alpha d^k p$	(zero prices for overproduced goods and zero intensities for inefficient firms)
(5)	$qBp > 0$	(something of value is produced)
(6)	$p,q \geq 0, \quad qf = ep = 1$	

where: m is the number of firms

n is the number of goods, $m \leq n$

α is the growth factor

q is an (1,m) vector of intensities

p is an (n,1) vector of relative prices

B is the (m,n) matrix of output coefficients.

The (i,j)-th entry represents the amount of the j-th good produced by the i-th firm when firm i is run at unit intensity level.

A^k is the (m,n) matrix of input coefficients.

The (i,j)-th entry reprsents the amount of the j-th good needed as input by the i-th firm when firm j is run at unit intensity level.

d^k is the $(1,n)$ vector of demand, where the i-th entry represents the public's demand from good i.

f is a m-dimensional vector whose entries are all equal to 1.

e is a n-dimensional row vector whose entries are all equal to 1.

It can be noted that in equilibrium the quantities from each good produced by a firm are determined but nothing is said about how the actual exchange of goods takes place between firms.

We want now to aggregate all the firms in m sectors, and the goods into products, such that one sector produces only one product and this product is produced by no other sector. We do this in order to achieve a Leontief type model. We will assume that the number of sectors is equal to m and so the macroeconomy equilibrium output is given by m intensities, where the intensity of a sector has the same meaning with the intensity of a firm. Since each sector produces a distinct product we will have m products, each of which is characterized by a relative price. As mentioned before, this will be the only difference between the microeconomy and the macroeconomy and so, in equilibrium, the output and the prices of the macroeconomy are also going to grow at a constant rate.

Under these assumptions, the equilibrium in the macroeconomy will be summarized by the following relations (a vNL with final demand model):

(7) $q(I-\alpha A^{\ell}) \geq \alpha d^{\ell}$

(8) $(I-\alpha A^{\ell})p \leq \alpha f d^{\ell} p$

(9) $q(I-\alpha A^{\ell})p = \alpha d^{\ell}p$

(10) $qp > 0$

(11) $q,p \geq 0, \quad qf = pe = 1$

where: m is the number of industries and the number of products,

α is the growth factor,

q is the (1,m) vector of intensities of sectors,

p is the (m,1) vector of prices and products,

A^1 is the (m,m) matrix of input-output coefficients.

The (i,j)-th entry represents the output of the j-th sector used as input in the i-th sector when sector i is run at unit level intensity,

f is a m-dimensional column vector for whose entries are equal to one,

e is a m-dimensional row vector all whose entries are equal to one,

d^{ℓ} is the (1,m) vector of final demand for products.

It can be noted that in this model the answer to the question: "who supplies whom?" is clear because for each product there is only one supplier and all the other sectors are demanders.

Let us now introduce a formal definition for consistent aggregation:

Definition 1[13/] Let (α^k, q^k, p^k) be the solution of a KMT model (B, A^k) with final demand d^k, and let $(\alpha^{\ell}, q^{\ell}, p^{\ell})$ be the solution of a vNL model (I,A) with final demand d^{ℓ}. Then (I, A^{ℓ}) is a consistent aggregation-disaggregation of (B, A^k) iff it exists a matrix A such that for any $d_1^k \geq 0$ we have: $p^{\ell} = Ap^k$, $\alpha^{\ell} = \alpha^k$, $q^{\ell} = q^k$, $d^{\ell} = d^k A^g$, and for any $d^{\ell} \geq 0$ we have: $\alpha^k = \alpha^{\ell}$, $q^k = q^{\ell}$, $p^k = A^g p^{\ell}$, $d^k = d^{\ell}A$, where: A^g is the generalized-inverse of A, A and A^g are non-negative, and d_1^k is a row vector containing m entries of d^k.[14/]

It is clear that our definition for consistent aggregation is consistent with the standard definition for consistent aggregation. Namely, it says that for consistent aggregation there should be a fixed aggregation procedure (the matrices A, A^g) such that for any value of the final demand the forecasts of the vNL model are the same as the "true" equilibrium values given by the KMT model. It should be noted that under this definition it doesn't matter if the forecasts are made at the micro level or at the macro level.

II.2. The following theorem will give us the necessary and sufficient conditions for a consistent aggregation between the microeconomy and the macroevonomy.

<u>Theorem 1.</u>[15] Let (B,A^k) be a KMT model with final demand, and unique growth rate, and $B,A^k \geq 0$.[16] Then there is a vNL model with final demand (I,A^ℓ) which is a consistent aggregation-disaggregation of (B,A^k) iff:

(a) rank $(B) = m$

(b) $B^g \geq 0$

(c) $A^K B^g B = A^K$,

where: B^g is the generalized inverse of B, $A^K = A^k + fd^k$, f is a m-dimensional column vector whose entries are all equal to one.

<u>Proof</u>: Let (α,g,p^k) be the solution of the KMT model (B,A^k). If (B,A^k) <u>is a consistent aggregation</u> of (I,A^ℓ), it follows that:

$$\exists A:\quad d^k = d^\ell A,\quad p^k = A^g p^\ell,\quad A \geq 0,\quad A^g \geq 0.$$

Using this, the relations (1) - (6) can be rewritten in the form:

$$g(BA^g - \alpha A^K A^g) \geq 0$$

$$(BA^g - \alpha A^K A^g)p^k \leq 0$$

$$g(BA^g - \alpha A^K A^g)p^k = 0$$

$$gBA^g p^k > 0$$

$$g, p^{\ell} \geq 0$$

where: $A^K = A^k + fd^k$, f is a m-dimensional column vector whose entries are all equal to one. By assumption, the model $(BA^g, A^K A^g)$ is a vNL model, so $BA^g = I$. This implies that:

$$0 \leq A^g = B^g \text{ and rank } (B) = m,$$

and so we proved that the conditions (a) and (b) are necessary.[17] In order to prove that the condition (c) is also necessary, let (α, g, p^{ℓ}) be the solution of the vNL model (I, A^{ℓ}) which is a consistent aggregation of (B, A^k). From the definition of consistent aggregation it follows that:

$$\exists A: \; d^{\ell} = d^k A^g, \; p^{\ell} = Ap^k, \; A \geq 0, \; A^g \geq 0.$$

Using this, the relations (7) - (11) can be rewritten in the form:

$$g(A - \alpha A^L A) \geq 0$$

$$(A - \alpha A^L A)p^k \leq 0$$

$$g(A - \alpha A^L A)p^k = 0$$

$$gAp^k > 0$$

$$g, p^k \geq 0,$$

where: $A^L = A^{\ell} + fd^{\ell}$, f is a m-dimensional column vector whose entries are all equal to one. From the condition that (I, A^L) is a consistent aggregation-disaggregation of (B, A^K) for any $d_1^k \geq 0$ it follows that the models (B, A^K) and $(A, A^L A)$ coincide. We have assumed that the rank of B is given by the first m columns and d_1^k is a m-dimensional row vector whose entries are equal to the first m entries of d^k. That implies that:

$$B = A \quad \text{and} \quad A^K = A^L A.$$

From $A^K = A^L A$, and rank $(B) = m$, and $B = A$ one can show that:

$$A^K B^g = A^L \quad \text{and} \quad A^K B^g B = A^L B \ ,$$

and so $A^K B^g B = A^K$, which proves that condition (c) is necessary.

In order to prove the sufficiency part, let (α, g, p^k) be a solution of (B, A^K). Then from (1)-(6) one can show:

$$g(I - \alpha A^K B^g) \geq 0 \qquad \text{by (a)-(c)}$$

$$(I - \alpha A^K B^g) B p^k \leq 0 \quad \text{by (c)}$$

$$g(I - \alpha A^K B^g) B p^k = 0 \quad \text{by (c)}$$

$$g B p^k > 0 \ .$$

By making the substitutions $A^L = A^K B^g$, and $p^\ell = Bp^k$, and $d^\ell = d^k B^g$, it follows that for any $d^k \geq 0$, (α, g, Bp^k) is a solution of the vNL model $(I\ A^K B^g)$ because $B \geq 0$ implies that $p^\ell = Bp^k \geq 0$. Similarly, let (α, g, p^ℓ) be the solution of the vNL model $(I, A^K B^g)$. Then from (7) - (11) one can show that:

$$g(B - \alpha A^K) \geq 0 \qquad \text{by } B \geq 0, \ (c)$$

$$(B - \alpha A^K) B^g p^\ell \leq 0 \qquad \text{by (a)}$$

$$g(B - \alpha A^K) B^g p^\ell = 0 \qquad \text{by (a)}$$

$$g B B^g p^\ell > 0 \qquad \text{by (a)}$$

Let $p^k = B^g p^\ell$, $d^k = d^\ell B$. It follows that for any $d^\ell \geq 0$, $(\alpha, g, B^g p^\ell)$ is a solution of the KMT model (B, A^K), because $B^g \geq 0$ implies that $p^k = B^g p^\ell \geq 0$.

Q.E.D.

As a corollary of Theorem 1, one can prove very easily the form of the aggregated solutions.

Corollary 1. Under the hypothesis of Theorem 1 if we have consistent aggregation then the micromodel and the macromodel have the same rate of growth (α) and the same intensities (g) and the relation between the microprices (p^k) and the macroprices (p^ℓ) are as given by the formula:

$$p^\ell = \frac{1}{eBp^k} Bp^k, \quad e = (1, \ldots, 1) \in \mathbb{R}^{(1,m)} \quad .$$

The results show that the aggregated prices are going to be a weighted of the microprices, where the fixed weights are given by the matrix B of output coefficients.

In order to get an interpretation of Theorem 1, we have to look at the net exchange in the two economies, where by net exchange we will understand the exchange that takes place between firms (sectors) net of the own production of the firms (sectors). Then the net exchange in the KMT economy (E^K) is:

$$E^K = Q(B - \alpha A^K)$$

where: Q is a (m,m) scalar matrix whose on-diagonal entries are given by the intensities of the firms and A^K has the same meaning as in Theorem 1. The net exchange in the vNL economy (E^L) is:

$$E^L = Q(I - \alpha A^L)$$

where: Q is a (m,m) scalar matrix whose on-diagonal entries are given by the intensities of the sectors, $A^L = A^\ell + fd^\ell$, f is a m-dimensional column vector whose entries are all equal to one.

Using these notations, the consistent aggregation between the micromodel and the macromodel implies that $E^K = E^L B$ for any final demand. This formula shows that the output from each good produced by the economy (each column of B) is distributed from suppliers to demanders according to a linear rule (E^L) which is the same for each of the n goods produced in the microeconomy.

Before going to the next section let us summarize the results obtained in this section. Due to our insufficient knowledge of the demand side of the market, very strong assumptions have to be imposed in order to make the vNL net exchange consistent with KMT net exchange. Secondly, the aggregation function (in our case the matrix B) is given by the microeconomy and not exogenously imposed. Thirdly, the aggregation function is independent of the quantities demand, and it is unique for a given microeconomy.

III. The Demand Function Case

In the first part of this section we introduce an explicit demand function and we study the conditions for consistent aggregation in the redefined micromodel. In the second part we will discuss the new results obtained.

III.1. Let us now introduce the demand of the public for consumption goods and saving goods:

$$y = q(B-A^k),\ \text{where}\ q,\ B,\ A^k\ \text{have the same meaning as in (2)-(6).}$$

Since qB represents output, given inputs qA^k, it follows that $(qB-qA^k)$ is the production of goods that are in excess over the amount needed to maintain a stationary economy (including the reproduction of workers). In order to be consistent with the consumption-saving pattern implicitly existing in the KMT with final demand model, let us assume that the public consumes as much as it saves.[18/] Let us denote by c and s the consumption and capital accumulation (savings) coefficients, where c and s are such that:

$$c + s = 1, \quad 0 \leq c \leq 1, \quad 0 \leq s \leq 1 \quad .$$

Then our assumption regarding the consumption savings behavior of the public implies that:

$$c = a = \frac{1}{2} \, .$$

Under these circumstances, then the public demand for consumption goods (d^k), is given by the formula:

$$d^k(q) = \frac{1}{2} \, q(B - A^k) \quad .$$

By substituting $d^k(q)$ in (2) - (6) and by denoting:

$$A^K = \frac{1}{2} \, (B + A^k)$$

the equilibrium relations (2)-(6) can be rewritten as:

(2') $\quad q(B-\alpha A^K) \geq 0$

(3') $\quad (B-\alpha A^K)p \leq 0$

(4') $\quad q(B-\alpha A^K)p = 0$

(5') $\quad qBp > 0$

(6') $\quad q,p \geq 0, \quad qf = ep = 1$

This is the micro economy and as it can be seen, it takes the form of a standard KMT model.

We want now to aggregate the KMT economy to a vNL economy:

(7') $q(I-\alpha A^L) \geq 0$

(8') $(I-\alpha A^L)p \leq 0$

(9') $q(I-\alpha A^L)p = 0$

(10') $qp > 0$

(11') $q,p \geq 0, \quad qf = ep = 1$

where q, p, α, I, e, f, have the same meaning as in (7)-(11). As in II.2 we will require the aggregation to be such that: any vNL solution is an aggregation of the KMT solution, and some KMT solutions are a disaggregation of the vNL solutions.

<u>Definition 2</u>. Let (α^k, q^k, p^k) be the solution of a KMT model (B, A^K), and let $(\alpha^\ell, q^\ell, p^\ell)$ be the solution of a vNL model (I, A^L). Then (I, A^L) is a <u>consistent aggregation-disaggregation</u> of (B, A^K) iff there is a matrix A and a function if such that $\alpha^k = \alpha^\ell$, $q^k = q^\ell$, $p^k = A^g p^\ell$, $d^k = f(d^\ell)$ and $p^\ell = Ap^k$, $d^\ell = f^{-1}(d^k)$, where A^g is the generalized inverse of A, A and A^g are non-negative, and f^{-1} is the inverse function of f.

It should be noted in the above definition that in this case we impose a specific form only on the aggregation for the microvariables while letting the aggregation of the final demands to be performed by some unspecified function. Given these requirements the next theorem will give us some sufficient conditions for consistent aggregation-disaggregation.

<u>Theorem 2</u>. Let (B,A^K) be a KMT model with unique growth rate and $B,A^K \geq 0$.[16/]

If (B,A^K) is such that:

(a) rank $(B) = m$

(b) $B^g \geq 0$

(c) $A^K B^g B = A^K$,

then there is a vNL model (I,A^L) which is a consistent aggregation-disaggregation of (B,A^K).

<u>Proof</u>: It is a reiteration of the sufficiency part of the proof of Theorem 1.

It is obvious that, in this case, the necessary condition for consistent aggregation-disaggregation should be weaker than in the first case. In order to see that, let us look at an example. In Example 2 (pp. 19), the KMT model (B_2,A_2^K) satisfies the necessary conditions from Theorem 2, while (B_1,A_1^K) does not satisfy these conditions. However, both models have exactly the same set of solutions, so the vNL model (I,A_2^L) can be considered a consistent aggregation-disaggregation of the KMT model (B_1,A_1^K), in the sense of Definition 2. Then Theorem 3 will show that this is always the case, i.e. the conditions (a) - (c) from Theorem 2 are consistent with the conditions for the existence of an KMT model which does not satisfy these restrictions but it has the same solutions with the model that satisfies them.

<u>Theorem 3.</u> Let (B,A^K) be a KMT model with unique non-negative growth rate and $B,A^K \geq 0$. If (B,A^K) is such that it satisfies conditions (a) - (c) from Theorem 2, then there is at least one KMT model, (B_*,A_*^K) which has the following properties:

(a) (B_*,A_*^K) has exactly the same solutions with (B,A^K),

(b) (B, A^K_*) does not have the properties (a) -(c) from Theorem 2.

Proof. Remark 1. Let $M_i = B_i - \alpha A^K_i$, $i = 1,2$. If two KMT models with unique growth rate, (B_1, A^K_1) and (B_2, A^K_2): $M_1 = M_2$ then (B_1, A^K_1) and (B_2, A^K_2) have exactly the same solutions.[19/]

Remark 2. If (B, A^K): it satisfies (2') - (6') and $B, A^K \geq 0$ then the growth rate is different than zero.[20/]

Let α be the growth rate for the model (B, A^K). Define (B_*, A^K_*):

$$B_* = B + E, \quad A^K_* = A^K + \frac{1}{2} E, \quad E \in \mathbb{R}_+^{(m,n)}, \quad m \leq n.$$

It is very easy to show that:

- B_*, $A^K_* > 0$,
- the model (B_*, A^K_*) has solutions because $eB_* > 0$ and $A^K_* f > 0$, $e = (1,\ldots,1) \in \mathbb{R}^{(1,m)}$,

  $$f = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}(n,1),$$

- the model (B_*, A^K_*) has a unique growth rate because $B_* + A^K_* > 0$
- If α is the unique growth rate of (B, A^K) then α is also the unique growth rate for (B_*, A^K_*).

Moreover $B - \alpha A^K = B_* - \alpha A^K_*$ and so by Remark 1 it follows that (B, A^K) and (B_*, A^K_*) have exactly the same solutions. That proves point (a) of the theorem.

In order to prove point (b), we will take special cases for E, and one can show that (B_*, A^K_*) does not satisfy (a) - (c) from Theorem 2. In order to show that rank $(B_*) \neq m$ and that B^g_* contains negative entries take:

$$E = \begin{bmatrix} \epsilon & \dots\dots & \epsilon \\ - & - - - - - & - \\ \epsilon & \dots\dots & \epsilon \end{bmatrix} \in \mathbb{R}^{(m,n)}, \quad \epsilon > 0, \quad m \leq n,$$

In order to show that $A_*^K B_*^g B \neq A_*^K$ take:

$$E = [E_1 \mid \mathbf{0}], \quad E_1 = \begin{bmatrix} \epsilon_1 & 0 \dots & 0 \\ 0 & \epsilon_2 \dots & 0 \\ - - & - - - & - - \\ 0 & 0 \dots & \epsilon_m \end{bmatrix}, \quad \epsilon_i > 0, \quad i = 1,\dots,m,$$

$\mathbf{0}$ is a $(m-n,m)$ zero matrix.

Q.E.D.

The relation between the solutions of (B_*, A_*^K) and its vNL image are given by Corollary 3.

<u>Corollary 3</u>. Let (α^k, q^k, p^k) be the solution of (B_*, A_*^K), and let $(\alpha^\ell, q^\ell, p^\ell)$ be the solution of the vNL model $(I, A^K B^g)$. Then one can show that $(I, A^K B^g)$ is a consistent aggregation-disaggregation (in the sense of Definition 2) of (B_*, A_*^K) and:

- $\alpha^k = \alpha^\ell, \qquad q^k = q^\alpha$

- $p^\ell = \dfrac{1}{eBp^k} Bp^k, \quad e = (1,\dots,1) \in \mathbb{R}^{(1,m)}$.

It should be noted that in this case matrix B is the aggregation matrix rather than B_*.

An interpretation of these results is as follows: if the goods satisfy the conditions of complementarity from Theorem 2 then the consistent aggregation is possible. Due to the fact that in this case the demand for goods for consumption and savings is given by an explicit demand function, rather than by some unknown parameters as in Section II, the conditions of complementarity are no longer necessary. Moreover the proof of Theorem 3 gives us a way of constructing KMT models that do not satisfy the conditions (a) - (c) of Theorem 2 but have an vNL model (I,A^L) which realizes a consistent aggregation-disaggregation in the sense of Definition 2.

III.2. The question which arises now is, what interpretation can one attach to this type of aggregation. In Section II we saw that the net exchange between firms is equal to the disaggregated net exchange between sectors. This explanation is valid for this case too. The only difference will be that, in this case, the disaggregation function might not be endogenous to the micromodel. For an example of such a case, let us go back to Example 2 (pp. 19), and let us denote the net exchange in the KMT model (B_1,A_1^K) with E_1^K, and the net exchange in its vNL aggregated image (I,A_2^L) with E_2^L. Then we have the following relation:

$$E_1^K = E_2^L B_2 \quad .$$

This relation is similar to the relation obtained at the end of the last section with the only difference, that in this case, the disaggregation function (B_2) is not endogenous to our micromodel. In other words the actual net exchange in the economy (E_1^K) is consistent with a linear trading rule (E_2^L) if the economy behaves <u>as if</u> it produced output B_2 instead of the actual output (B_1).

Another property of this type of aggregation is the fact that the way we aggregate the goods into products is not unique. In order to show that let us go back to Example 1 (pp. 18), and let us assume that our micromodel is the KMT model (B_2, A_2^K). Then we can say that (I, A_2^L) or (I, A_3^L) are the corresponding aggregated vNL models. Consequently the prices of the products can be either p_2^L or p_3^L.

A last remark is concerned with the robustness of the results obtained in this section. Namely we want to see if the aggregation function depends on changes in the demand. Since in this case the demand is given by a function, the only changes that can occur in the quantity demanded in equilibrium are due to changes in the coefficient of the demand function. At this moment we do not have definite results for this problem but it seems that for a very large set of cases the aggregation procedure is not affected by changes in the coefficient of the demand function.

IV. Discussion

In this section we will review the results obtained in the last two sections.

Though we used a general equilibrium model as a framework for studying how heterogenous goods can be aggregated into a homogeneous good, the results obtained in Section 2 are similar to the results obtained for other linear models. As a matter of fact, though we considered only non-negative values for the parameters, the conditions for consistent aggregation are of the same type of Klein's results.[21/] Because of that, the aggregation function in our case has similar characteristics: it is independent of the values of the parameters it is unique and it is endogenous for a given micromodel. As a consequence the macromodel is unique and it can be derived directly from the micromodel once the form of the macromodel is given.

The different results obtained in Section III are due to two factors: on one hand we allowed the prices and the demand to be aggregated in different ways, and on the other hand we introduced a demand function instead of the vector of parametric demand. In that case the aggregation function is not unique and it is not endogenous to the micromodel. As a consequence, the macromodel in itself is less connected with the micromodel than in the first case. Though we do not have definite results about the robustness of the aggregation function to changes in the coefficients of the demand function, we are quite confident that the class of changes that does not affect the aggregation function is quite general.

V. Concluding Remarks

As mentioned in the introduction, one way of explaining how heterogenous goods can be aggregated into a homogenous one is based on properties of complementarity, substitutability and similarities between firms. Casual empirical evidence seems to show that models which assume a high degree of homogeneity for the output of a firm or group of firms performed quite well despite the heterogeneity existing in the "real world". [22/]

In the second section of the paper we performed such an aggregation in a general equilibrium framework with parametric demand, and we showed that the necessary and sufficient conditions for consistent aggregation are based on conditions similar to those mentioned above. It should be noted though, that these conditions imply a potentially observable structure of the trading that takes place among the firms of the micromodel. So, if one observes that the trading between firms has a linear form, then one can look for an aggregation procedure as required by our Definition 1.

Our approach to the aggregation problem, as presented in Section III, is based on two departures from the standard approach: on one hand we allowed the microvariables of model (the prices) and microparameters (the vector of demand) to be aggregated in different ways, and on the other hand we introduced a demand function. The fact that the microvariables and the microparameters are usually required to be aggregated in the same way is more a matter of convenience than one derived from the micromodel.[23/]

The introduction of the demand function is motivated by our intent to approach the aggregation problem as the problem of existence of an equilibrium in a general equilibrium model.

The results of our approach show that the aggregation procedure is dependent on changes of the parameters of the demand function. This conclusion differs from the results obtained through the Theil-Peston approach [23], [21]. It seems that the difference comes from the fact that the problem we study is somewhat different from the one they analyzed: we assumed that the macrovariables do not appear among the variables of the micromodel. It can be argued that this is the case in many instances when an aggregation is performed.

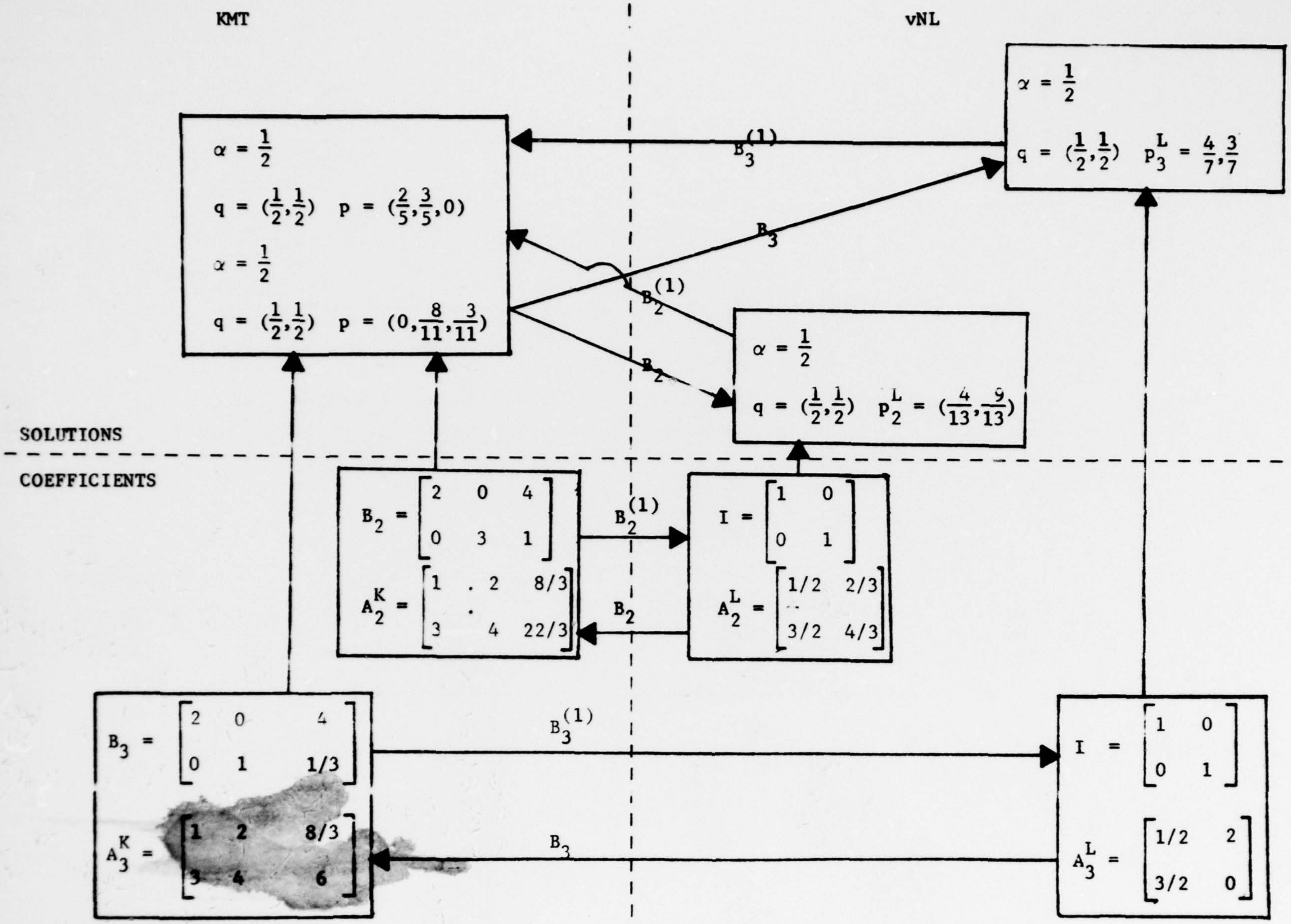
In obtaining our results, we made strong assumptions about the equilibrium conditions in the micromodel. It is obvious that a more general answer to this problem should come from a less structured equilibrium model for the microeconomy. In what follows we will outline some directions for further study.

One direction is connected with the undeterminancy of the aggregation procedure we mentioned above. This would suggest that there are other factors which we didn't take into consideration when aggregating the heterogenous goods into a homogeneous one.[24/]

Another direction should be the study of the robustness of the aggregation procedure to changes in the demand, matter which seems to be a stability type of question.

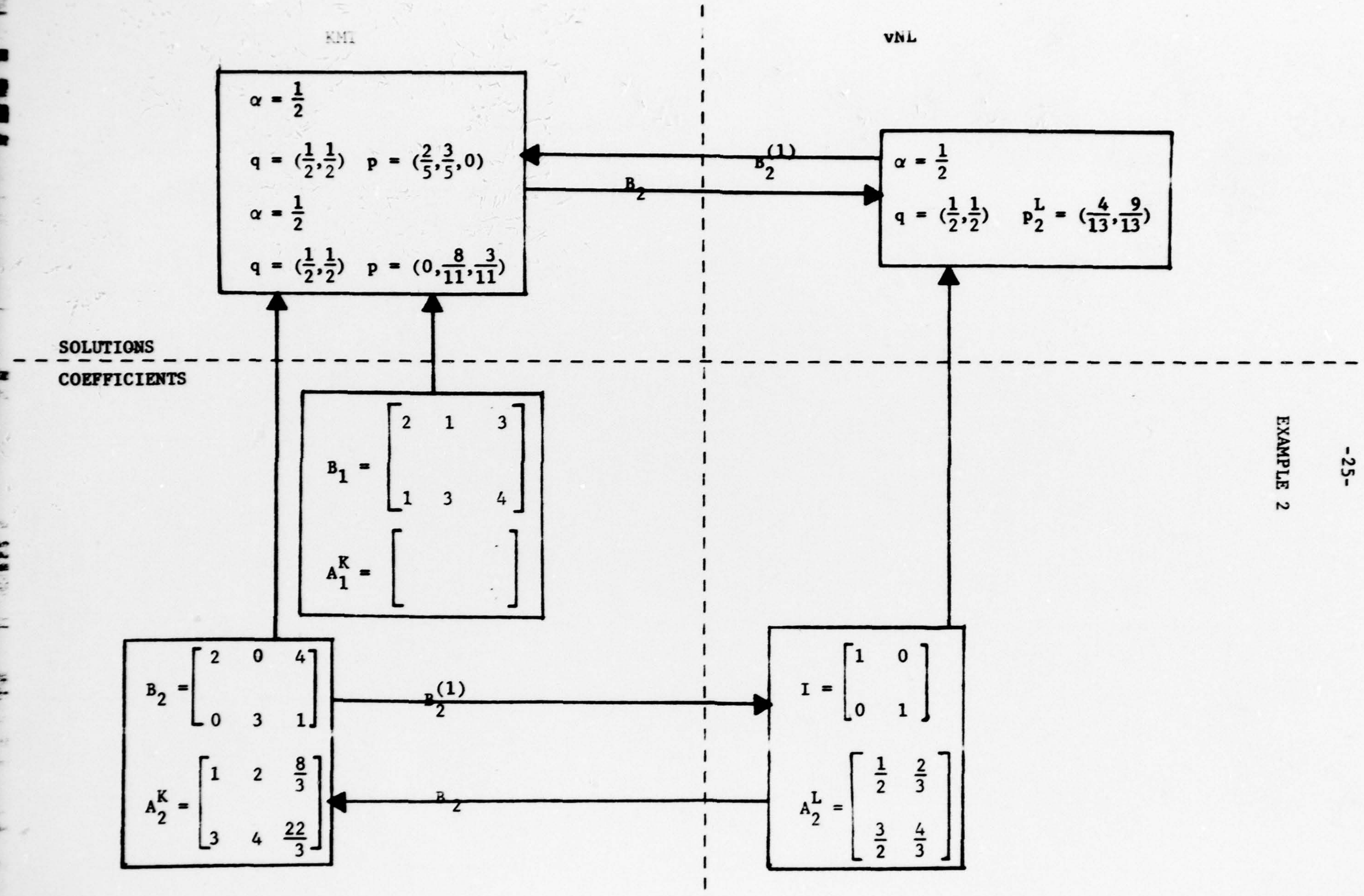
Another point of interest is the fact that we had to impose exogenously the form of aggregated model. As Grunfeld and Griliches suggested it, the amount of knowledge about the microeconomy seemed to be a very important factor in deciding upon the form of the macromodel for a given micromodel.[25] So in this case again it seems the problems connected with aggregation are largely due to our incomplete knowledge of the microeconomy.

EXAMPLE 1



where: $B_i^{(1)}$ = the generalized inverse of B_i, $i = 1,2$

EXAMPLE 2



where $B_2^{(1)}$ = the generalized inverse of B_2

Footnotes

1. In general the objects of the Aggregation Theory are: a microsystem, a macrosystem (which is in general simplier), and a aggregation function. Then an aggregation is consistent (good) if there is no loss of information working with the macrosystem rather than using the microsystem. For a more formal approach see Ijiri [5, pp. 766-769].

   For the assumptions underlying the Leontief model a good reference is Malinvaud [14, pp. 189-pp. 194]. Other references Leontief [11], [12].

2. Here we adopted Intriligator's term for this type of production function [6, pp. 187-189].

3. [8, pp. 94-95].

4. Klein's rules are an attempt to "transfer" all the characteristics of the micromodel to macromodel and not only the numerical value of the solutions.

5. Good general references for this field are [1], [13] and [16]. For a presentation of the KMT model see [7].

6. See [7, pp. 129-132] for a study of aggregation in a standard KMT model.

7. The efficiency properties for some models of the von Neumann variety were proved by Truchon [26].

8. [16, pp. 204].

9. [16, pp. 203-209].

10. For a review of the literature in this field see [15] and [26, pp. 12-25].

11. [7, pp. 117-118].

12. An analysis of the implicit assumptions of the case with parametric demand can be found in [26, pp. 12-15].

13. As it can be seen our definition is a combination between the standard definition [25, pp. 517-518], and Fisher's definition [3, pp. 8-12].

14. The concept of generalized inverse and some of its properties are in [18], [20].

15. The proof of this theorem, is partly based on the material from one of our earlier papers.

16. The sufficient conditions for a KMT model (B, A^k) to have a unique growth rate are that: $B + A^k > 0$. See [14, pp. 3].

17. See [2, pp. 11-12, Lemma 2].

18. [15, pp. 118-121].

19. It follows from the computational procedure used to find (α_i, q_i, p_i), $i = 1,2$. See [16, pp. 53-54].

20. The conditions for the existence of solutions for a KMT model with non-negative matrices requires that $eB > 0$ and $A^K f > 0$, $e = (1,\ldots,1) \in \mathbb{R}^{(1,m)}$, $f = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^{(n,1)}$. *These conditions are implicitly assumed to be satisfied* in Theorem 3 and they imply that the growth rate must be different than zero.

21. [9, pp. 310-312].

22. For an evaluation of an application of input-output analysis to the Dutch economy see [22], [24].

23. A similar result can be found in [10].

24. For the case in which $m \geq 2$, it seems that one way of solving this problem is to require that the relative weight of the actual revenue of a firm to be equal to the relative weight of the actual revenue of the corresponding sector in the macromodel, where the actual revenue of firm i is $q_i \underline{b}_i p^k$, $\underline{b}_i$ is the i-th row of matrix B.

25. [4, pp. 10].

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The paper presents the aggregation of heterogenous production into a homogenous one in an economy described by a General Equilibrium model of the von Neumann type (a KMT model and final demand). The resulting aggregated model will be of the von Neumann-Leontief type. ~~We will consider~~ two cases will be considered, one in which the final demand is given by a vector of parameters, and another case ~~in~~ which ~~we~~ will have a final demand function.

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